Exercise 7.8.1

Consider the Riccati equation $y' = y^2 - y - 2$. A particular solution to this equation is $y = 2$. Find a more general solution.

Solution

Since one solution is known, the Riccati equation can be transformed into a Bernoulli equation by making the substitution $y = 2 + u$. Differentiating both sides of the substitution with respect to x gives $y' = u'$.

$$y' = y^2 - y - 2 \quad \rightarrow \quad u' = (2+u)^2 - (2+u) - 2 \quad \rightarrow \quad u' = u^2 + 3u$$

Bring $3u$ to the left side.

$$u' - 3u = u^2$$

Divide both sides by u^2.

$$u^{-2}u' - 3u^{-1} = 1$$

Make the substitution $p = u^{-1}$. Then $p' = -u^{-2}u'$ by the chain rule.

$$-p' - 3p = 1$$

Multiply both sides by -1.

$$p' + 3p = -1$$

This is a first-order linear ODE that can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^x 3\, ds\right) = e^{3x}$$

Proceed with the multiplication.

$$e^{3x}p' + 3e^{3x}p = -e^{3x}$$

The left side can be written as $d/dx(Ip)$ by the product rule.

$$\frac{d}{dx}(e^{3x}p) = -e^{3x}$$

Integrate both sides with respect to x.

$$e^{3x}p = -\frac{1}{3}e^{3x} + C$$

Divide both sides by e^{3x}.

$$p(x) = -\frac{1}{3} + Ce^{-3x}$$

Now that the ODE is solved, change back to u.

$$u^{-1} = -\frac{1}{3} + Ce^{-3x}$$

Invert both sides.

$$u(x) = \frac{1}{-\frac{1}{3} + Ce^{-3x}} \cdot \frac{3e^{3x}}{3e^{3x}} = \frac{3e^{3x}}{3C - e^{3x}}$$

Therefore, since $y = 2 + u$,

$$y(x) = 2 + \frac{3e^{3x}}{3C - e^{3x}}.$$